Uniform Turán density—palette classification

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On the last episode...

Question (Turán's Tetrahedron Problem)

What is the maximum edge density of a 3-graph on n vertices that does not contain a tetrahedron ($K_4^{(3)}$) as a subgraph?

Known

$\frac{5}{9} \leq \pi(K_4^{(3)}) \leq 0.5615$ *(Razborov; Baber, Talbot)*

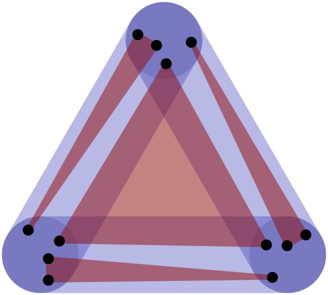

Figure: Turán's construction with density $\frac{5}{9}$

Definition

For $d \in [0,1]$ and $\eta > 0$ we say that 3-graph F is (d,η)-dense if for all $U \subseteq V$ the following inequality holds:

$$\left| \binom{U}{3} \cap E \right| \geq d \binom{|U|}{3} - \eta |V|^3.$$

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Definition

Let F be a 3-graph. We define its *uniform Turán density* to be

$$\pi_u(F) = \sup \Big\{ d \in [0,1] : \text{for every } \eta > 0 \text{ and } n \in \mathbb{N}, \text{ there exists an } F\text{-free } (d,\eta)\text{-dense 3-graph } H \text{ of order at least } n \Big\}.$$

Definition

Let P be a finite set of colors. We call a subset $\mathcal{P} \subseteq P^3$ a coloring palette.

Example

Let $P = \{ \textit{red}, \textit{green}, \textit{blue} \}$ and $\mathcal{P} = \{ (\textit{red}, \textit{green}, \textit{blue}) \}$

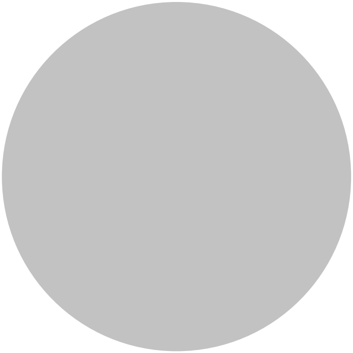
Definition

We say that 3-graph F is $\mathcal{P}$-colorable if there exists an ordering $\prec$ of the vertex set of F and an assignment $\varphi : \partial F \to P$ with the property that for all $uvw \in E(F)$ with $u \prec v \prec w$ it holds that

$$(\varphi(uv), \varphi(uw), \varphi(vw)) \in \mathcal{P}.$$

Proposition

Let F be a 3-graph and $\mathcal{L}$ be a palette. If F is not $\mathcal{L}$-colorable, then $\pi_u(F) \geq \lambda(\mathcal{L})$.



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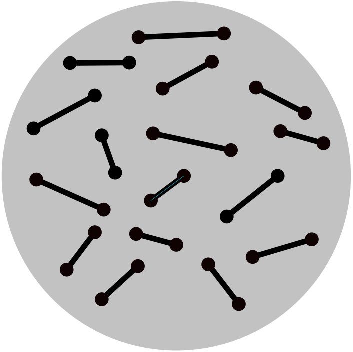

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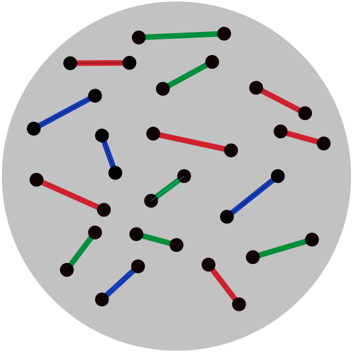

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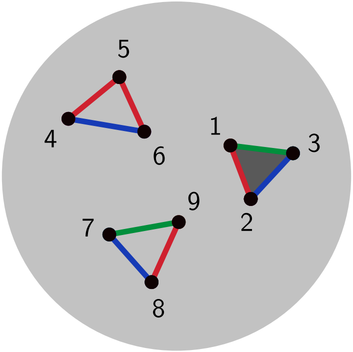

Theorem (Lamaison 2024)

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Definition

Let $\mathcal{P}$ and $\mathcal{R}$ be two palettes. We say that a color map $\psi : P \to R$ is a *palette homomorphism* if for every $p_1, p_2, p_3 \in P$ it holds that

$$(p_1, p_2, p_3) \in \mathcal{P} \implies (\psi(p_1), \psi(p_2), \psi(p_3)) \in \mathcal{R}.$$

Proposition

Let $\mathcal{U}$ be a palette such that for every palette $\mathcal{P}$ with $\lambda(\mathcal{P}) \geq d$ there is a homomorphism from $\mathcal{U}$ to $\mathcal{P}$ or to inv$(\mathcal{P})$. If 3-graph F is $\mathcal{U}$-colorable, then $\pi_u(F) \leq d$.

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Proof.

Let $\mathcal{P}$ be palette by which F is not colorable and $\pi_u(F) > d$, then there is a homomorphism $\psi : \mathcal{U} \to \mathcal{P}$ and coloring $\varphi : \partial F \to \mathcal{U}$. But $\psi\varphi : \partial F \to \mathcal{P}$ is a $\mathcal{P}$-coloring of F. □

Theorem (Reiher, Rödl, Schacht 2017)

There is no 3-graph F with $\pi_u(F) \in (0, 1/27)$.

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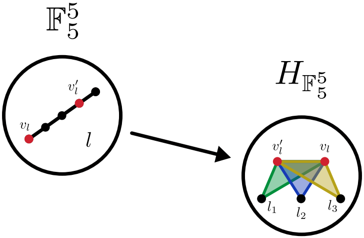
Theorem (Garbe, Ilkovič, Kráľ, K., Lamaison 2024+)

There exists a 3-graph F with $\pi_u(F) = 8/27$.



Theorem (Kráľ, K., Lamaison, Tardos 2025+)

Let $\mathcal{L}$ and $\mathcal{U}$ be two palettes. There exists a 3-graph H that is $\mathcal{U}$-colorable but not $\mathcal{L}$-colorable if and only if there is no homomorphism from the palette $\mathcal{U}$ to the palette $\mathcal{L}$ or $\mathrm{inv}(\mathcal{L})$.

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Theorem (King, Piga, Sales, Schülke 2025+)

For every $\mathcal{P}$, there is a finite family $\mathcal{F}$ of 3-graphs with $\pi_u(\mathcal{F}) = \lambda(\mathcal{P})$.

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Proposition

Let $\mathcal{U}_1, \ldots, \mathcal{U}_k$ *be a collection of palettes such that for every palette* $\mathcal{P}$ *with* $\lambda(\mathcal{P}) \geq d$ *there is for some* $i \in [k]$ *a homomorphism from* $\mathcal{U}_i$ *to* $\mathcal{P}$ *or* $inv(\mathcal{P})$*. If 3-graph* H *is* $\mathcal{U}_i$*-colorable for every* $i \in [k]$*, then* $\pi_u(H) \leq d$*.*

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Theorem (Kráľ, K., Lamaison, Tardos 2025+)

Let $\mathcal{U}_1, \dots, \mathcal{U}_k$ and $\mathcal{L}$ be palettes. There exists a 3-graph H that is $\mathcal{U}_1$-colorable, ..., $\mathcal{U}_k$-colorable but not $\mathcal{L}$-colorable if and only if for every $i \in [k]$ there is no homomorphism from $\mathcal{U}_i \times \prod_{j \neq i} \mathcal{U}_j^{(s)}$ to $\mathcal{L}$ or $inv(\mathcal{L})$.

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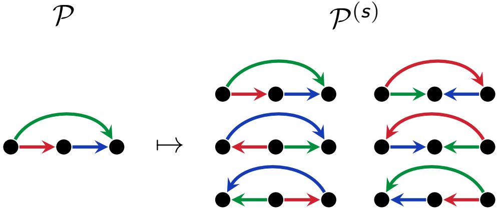

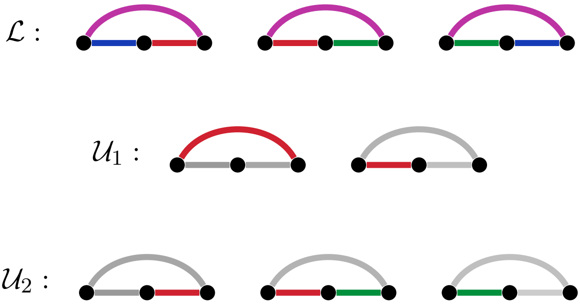

$\mathcal{L}$:
$\mathcal{U}_1$:
$\mathcal{U}_2$:

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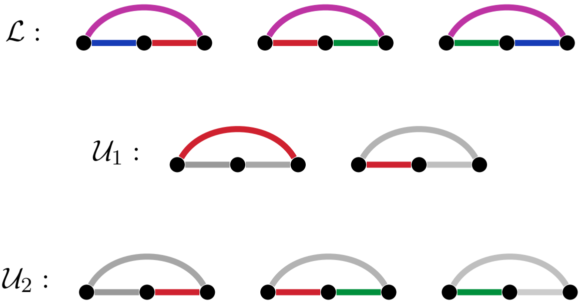
1. Choice of $p({\color{magenta}\bullet}) = 1/3$, $p({\color{red}\bullet}) = p({\color{green}\bullet}) = p({\color{blue}\bullet}) = 2/9$ for palette $\mathcal{L}$ gives $\lambda(\mathcal{L}) \geq 4/81$.



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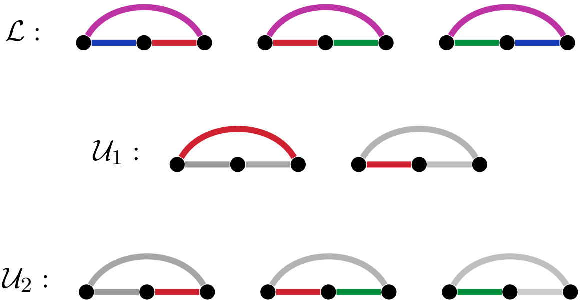

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2. For every $\mathcal{P}$ with $\lambda(\mathcal{P}) \geq 4/81$, we have a homomorphism from $\mathcal{U}_1$ or from inv$(\mathcal{U}_2)$ to $\mathcal{P}$ or inv$(\mathcal{P})$.
3. There is no homomorphism from $\mathcal{U}_1 \times \mathcal{U}_2^{(s)}$ or from $\mathcal{U}_1^{(s)} \times \mathcal{U}_2$ to $\mathcal{L}$.

There exists a 3-graph F with $\pi_u(F) = 4/81$.

Theorem (Lamaison, Wu 2025++)

For every integer d, there exists a 3-graph F with $\pi_u(F) = \alpha$, where the minimal polynomial of α has degree at least d.

Thank you for your attention!

Theorem (Kráľ, K., Lamaison, Tardos 2025+)

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As there is a homomorphism $\mathcal{U} \to \mathcal{L}$, every $\mathcal{U}$-colorable 3-graph is also $\mathcal{L}$-colorable.

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2. If every $\mathcal{U}$-colorable graph is $\mathcal{L}$-colorable, then either every ordered $\mathcal{U}$-colorable 3-graph is $\mathcal{L}$-colorable or every ordered $\mathcal{U}$-colorable 3-graph is inv$(\mathcal{L})$-colorable.

Lemma

If every ordered $\mathcal{U}$-colorable 3-graph is also $\mathcal{L}$-colorable, there is a homomorphism $\mathcal{U} \to \mathcal{L}$.

1. Create random ordered $\mathcal{U}$-colorable graph H which is $\mathcal{L}$-colorable by assumption.

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5. Use Ramsey's Theorem to obtain a tripartite graph where all color mappings agree.
6. Use regularity of the tripartite graphs to conclude that the color mapping is a homomorphism.